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Fixed point and approximation results for multimaps in S-KKM class

Naseer Shahzad[a,b,*]

[a] Department of Mathematics, Faculty of Science, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[b] The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

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Abstract

The paper discusses new fixed point and approximation theorems for multimaps in the class S-KKM.

Keywords: Fixed point; Condensing multimap; SKKM property; Approximation

1. Introduction

In 1969, Ky Fan [5] established the following results:

Let C be a nonempty, compact, convex subset of a normed space E. Then for any continuous mapping f from C to E, there exists an $x_0 \in C$ with

$$\|x_0 - f(x_0)\| = \inf_{y\in C} \|f(x_0) - y\|.$$

This result has been generalized to other sets C and other types of maps; see, for instance, [1,6,8–14,16,17]. Recently, Lin and Park [10] obtained a multivalued version of Ky Fan's result for α-condensing $\mathscr{U}_c^\kappa$ maps (see definition below) defined on a closed ball in a Banach space. More recently, O'Regan and Shahzad [13] extended their result to countably condensing maps. The aim of this paper is to obtain some Ky

* Corresponding author. Department of Mathematics, Faculty of Science, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail address: naseer_shahzad@hotmail.com (N. Shahzad).

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Fan approximation type results for Φ-condensing and 1-Φ-contractive s-KKM(C,C,E) multimaps, where C is closed convex subset of a Hausdorff locally convex space E with $int(C)\neq\emptyset$. Since every α-condensing map $F:C\to 2^E$ is Φ-condensing if C is complete and since $\mathscr{U}_c^{\kappa}$ class is a subclass of the s-KKM class, our results generalize the work of Lin and Park [10]. We also derive the Leray–Schauder-type result of Chang et al. [2] as an application of our approximation result.

2. Preliminaries

Let E be a Hausdorff locally convex space. For a nonempty set $Y\subseteq E$, 2^Y denotes the family of nonempty subsets of Y. If L is a lattice with a minimal element 0, a mapping $\Phi:2^E\to L$ is called a generalized measure of noncompactness, provided the following conditions hold:

(a) $\Phi(A)=0$ if and only if $\bar{A}$ is compact.
(b) $\Phi(\overline{co}(A))=\Phi(A)$; here $\overline{co}(A)$ denotes the closed convex hull of A.
(c) $\Phi(A\cup B)=\max\{\Phi(A),\Phi(B)\}$.

It follows that if $A\subseteq B$, then $\Phi(A)\leqslant\Phi(B)$. Let C be a nonempty subset of a Banach space X. The Kuratowskii measure of noncompactness is the map $\alpha:2^X\to L$ defined by

$$\alpha(A)=\inf\{\varepsilon>0: A \text{ can be covered by a finite number of sets each of diamter less than } \varepsilon\}$$

for $A\in 2^X$. The Hausdorff measure of noncompactness is the map $\chi:2^X\to L$ defined by

$$\chi(A)=\inf\{\varepsilon>0: A \text{ can be covered by a finite number of balls with radius less than } \varepsilon\}$$

for $A\in 2^X$. Examples of the generalized measure of noncompactness are the Kuratowskii measure and the Hausdorff measure of noncompactness (see [15]).

Let C be a nonempty subset of a Hausdorff locally convex space E and $F:C\to 2^E$. Then F is called Φ-condensing provided that $\Phi(A)=0$ for any $A\subseteq C$ with $\Phi(F(A))\geqslant\Phi(A)$. It is clear that a compact mapping is Φ-condensing and also every mapping defined on a compact set is necessarily Φ-condensing. Suppose that L is a lattice with a minimal element 0 and that for each $l\in L$ and $\lambda\in\mathbf{R}$, with $\lambda>0$, an element $\lambda l\in L$ is defined. A mapping $F:C\to 2^E$ is called a k-Φ-contractive map ($k\in\mathbf{R}$ with $k>0$) provided that $\Phi(F(A))\leqslant k\Phi(A)$ for each $A\subseteq C$ and $F(C)$ is bounded. Obviously, if C is complete, F is k-Φ-contractive, with $0<k<1$, and $\Phi=\alpha$ or χ, then F is Φ-condensing.

Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2, respectively. Let $F:X\to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say F is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now F is *acyclic* if F is upper semicontinuous with acyclic values. The map F is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2, respectively, a (U,V)-approximate continuous selection of $F: X \to K(Y)$ is a continuous function $s: X \to Y$ satisfying

$$s(x) \in (F[(x+U)\cap X] + V) \cap Y \quad \text{for every } x \in X.$$

We say F is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2, respectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p: Y \to X$ is called a Vietoris map if the following two conditions hold:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Definition 2.1. A multifunction $\phi: X \to K(Y)$ is *admissible* (strongly) in the sense of Gorniewicz, if $\phi: X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p: Z \to X$ and $q: Z \to Y$ such that

(i) p is a Vietoris map and
(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

Remark 2.1. It should be noted [7, p. 179] that ϕ upper semicontinuous is superfluous in Definition 2.1.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X,Y)$ denotes the set of maps $F: X \to 2^Y$ belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. A class $\mathscr{U}$ of maps is defined by the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single valued continuous functions;
(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued; and
(iii) for any polytope P, $F \in \mathscr{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathscr{U}$.

Definition 2.2. $F \in \mathscr{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathscr{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathscr{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Gorniewicz.

Let Q be a subset of a Hausdorff topological space X. We let $\bar{Q}$ (respectively, $\partial(Q)$, $int(Q)$) to denote the closure (respectively, boundary, interior) of Q.

Let C be a subset of a Hausdorff topological vector space E and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) = \{x + r(y - x):\ y \in C, r \geqslant 0\}.$$

If C is convex and $x \in C$, then

$$I_C(x) = x + \{r(y - x):\ y \in C, r \geqslant 1\}.$$

Definition 2.3. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family

$$\{\overline{S(x)}:\ x \in X\}$$

has the finite intersection property. We let

$$KKM(X, Y) = \{T : X \to 2^Y : T \text{ has the KKM property}\}.$$

Remark 2.2. If X is a convex space, then $\mathscr{U}_c^{\kappa}(X, Y) \subset KKM(X, Y)$ (see [4]).

Definition 2.4. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized S-KKM map w.r.t. T. If the map $T : X \to 2^Z$ satisfies that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then F is said to have the S-KKM property. The class

$$S\text{-}KKM(X, Y, Z) = \{T : Y \to 2^Z : T \text{ has the S-KKM property}\}.$$

Remark 2.3. If $X = Y$ and S is the identity mapping $\mathbf{1}_X$, then $S\text{-}KKM(X, Y, Z) = KKM(X, Z)$. Also $KKM(Y, Z)$ is a proper subset of $S\text{-}KKM(X, Y, Z)$ for any $S : X \to 2^Y$ and so $S\text{-}KKM(X, Y, Z)$ is a very large class of maps which includes other important classes of multimaps (see [2,3] for examples).

Remark 2.4. Let X be a convex subset of a Hausdorff topological space, Y a convex space, and Z, W topological spaces and $S : X \to 2^Y$. If $F : S\text{-}KKM(X, Y, Z)$ and $f \in \mathscr{C}(Z, W)$, then $f \circ F \in S\text{-}KKM(X, Y, W)$ (see [3]).

The following result [2] will be needed in the sequel. Throughout the paper, we shall assume that $f \circ F$ is closed whenever f is continuous and F is closed.

Lemma 2.1. *Let C be a nonempty, closed, convex subset of a Hausdorff locally convex space E Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, C)$ is a closed Φ-condensing map. Then F has a fixed point in C.*

3. Main results

Theorem 3.1. *Let Φ be either α or χ and C a nonempty, closed, convex subset of a Banach space E. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, C)$ is a closed 1-Φ-contractive map. In addition, assume the following condition holds:*

$$\begin{cases} if\ \{x_n\} \subseteq C\ with\ y_n \in F(x_n)\ for\ all\ n\ and\ x_n - y_n \to 0 \\ as\ n \to \infty,\ then\ there\ exists\ an\ x_0 \in C\ with\ x_0 \in F(x_0). \end{cases}$$

Then F has a fixed point in C.

Proof. Fix $v \in C$. For each n, define F_n by

$$F_n(x) = \lambda_n v + (1 - \lambda_n)F(x),$$

where $\{\lambda_n\} \subseteq (0, 1)$ with $\lambda_n \to 0$ as $n \to \infty$. Consider the mapping $g_n(y) = \lambda_n v + (1 - \lambda_n)y$. Then each g_n is continuous. Since $F \in s\text{-}KKM(C, C, C)$, by Remark 2.4, $F_n = g_n \circ F \in s\text{-}KKM(C, C, C)$. Since F is closed and g_n is continuous, each $F_n = g_n \circ F$ is closed. Also, each F_n is $(1 - \lambda_n)$-Φ-contractive and so is Φ-condensing. By Lemma 2.1, each F_n has a fixed point $x_n \in C$, i.e., $x_n \in \lambda_n u + (1 - \lambda_n)F(x_n)$ for each n. Choose $y_n \in F(x_n)$ with $x_n = \lambda_n u + (1 - \lambda_n)y_n$. It further implies that $x_n - y_n = \lambda_n(u - y_n) \to 0$ as $F(C)$ is bounded. By hypothesis, there exists an $x_0 \in C$ with $x_0 \in F(x_0)$. $\square$

Let C be a convex subset of a Hausdorff locally convex space E with $0 \in int(C)$. The Minkowski functional p of C is defined by

$$p(x) = \inf\{r > 0 : x \in rC\}.$$

The following properties of the Minkowski functional are well known:

(i) p is continuous on E;
(ii) $p(x + y) \leqslant p(x) + p(y),\ x, y \in E$;
(iii) $p(\lambda x) = \lambda p(x),\ \lambda \geqslant 0,\ x \in E$;
(iv) $0 \leqslant p(x) < 1$, if $x \in int(C)$;
(v) $p(x) > 1$, if$x \notin \bar{C}$;
(vi) $p(x) = 1$, if $x \in \partial C$.

For $x \in E$, set

$$d_p(x, C) = \inf\{p(x - y) : y \in C\}.$$

Theorem 3.2. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in C$ and U a convex open neighborhood of 0. Suppose $s : \bar{U} \cap C \to \bar{U} \cap C$*

is surjective and $F \in s\text{-}KKM(\bar{U} \cap C, \bar{U} \cap C, C)$ *is a closed* Φ*-condensing map. Then there exist* $x_0 \in \bar{U} \cap C$ *and* $y_0 \in F(x_0)$ *with*

$$p(y_0 - x_0) = d_p(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C),$$

where p *is the Minkowski functional of* U. *More precisely, either* (i) F *has a fixed point* $x_0 \in \bar{U} \cap C$, *or* (ii) *there exist* $x_0 \in \partial_C(U)$ *and* $y_0 \in F(x_0)$ *with*

$$0 < p(y_0 - x_0) = d_p(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C),$$

where $\partial_C(U)$ *denotes the boundary of* U *relative to* C.

Proof. Let $r : E \to \bar{U}$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in \bar{U}, \\ \dfrac{x}{p(x)} & \text{if } x \notin \bar{U}, \end{cases}$$

that is,

$$r(x) = \frac{x}{\max\{1, p(x)\}} \quad \text{for } x \in E.$$

Since $0 \in U = int(U)$, p is continuous and so r is continuous. Let f be the restriction of r to C. Since C is convex and $0 \in C$, it follows that $f(C) \subseteq \bar{U} \cap C$. Also $f \in \mathscr{C}(C, \bar{U} \cap C)$. By Remark 2.4, $f \circ F \in s\text{-}KKM(\bar{U} \cap C, \bar{U} \cap C, \bar{U} \cap C)$. Set $G = f \circ F$. Then G is closed. We now show that G is Φ-condensing. Let A be a subset of $\bar{U} \cap C$ such that $\Phi(A) \leqslant \Phi(G(A))$. If $F(A) \subseteq \bar{U} \cap C$, then $G(A) \subseteq F(A)$ and so $\Phi(A) \leqslant \Phi(G(A)) \leqslant \Phi(F(A))$. This implies that $\bar{A}$ is compact since F is Φ-condensing. On the other hand, if $F(A) \subseteq C \setminus \bar{U}$, then $G(A) \subseteq co(\{0\} \cup F(A))$ and so

$$\begin{aligned} \Phi(A) \leqslant \Phi(G(A)) &\leqslant \Phi(co(\{0\} \cup F(A))) \\ &\leqslant \Phi(\{0\} \cup F(A)) \\ &= \max\{\Phi(\{0\}), \Phi(F(A))\} = \Phi(F(A)), \end{aligned}$$

which gives $\bar{A}$ is compact. As a result, G is Φ-condensing. Now Lemma 2.1 guarantees that G has a fixed point i.e. there exists an $x_0 \in \bar{U} \cap C$ with $x_0 \in G(x_0)$. Then there exists some $y_0 \in F(x_0)$ with $x_0 = f(y_0)$. We now consider two cases: (i) $y_0 \in \bar{U} \cap C$ or (ii) $y_0 \in C \setminus \bar{U}$.

(i) Suppose $y_0 \in \bar{U} \cap C$. Then $x_0 = f(y_0) = y_0$. Consequently,

$$p(y_0 - x_0) = 0 = d_p(y_0, \bar{U} \cap C)$$

and x_0 is a fixed point of F. On the other hand, if $y_0 \in C \setminus \bar{U}$, then

$$x_0 = f(y_0) = \frac{y_0}{p(y_0)}.$$

As a result, for any $x\in\bar U\cap C$,

$$\begin{aligned} p(y_0-x_0) &= p\left(y_0-\frac{y_0}{p(y_0)}\right)=\left(\frac{p(y_0)-1}{p(y_0)}\right)p(y_0)\\ &= p(y_0)-1\leqslant p(y_0)-p(x)=p((y_0-x)+x)-p(x)\\ &\leqslant p(y_0-x). \end{aligned}$$

This implies that

$$p(y_0-x_0)=\inf\{p(y_0-z): z\in\bar U\cap C\}=d_p(y_0,\bar U\cap C).$$

Moreover, $p(y_0-x_0)>0$ since $p(y_0-x_0)=p(y_0)-1$.

Let $z\in I_{\bar U}(x_0)\cap C\setminus(\bar U\cap C)$. Then there exists $y\in\bar U$ and $c\geqslant 1$ with $z=x_0+c(y-x_0)$. Suppose that

$$p(y_0-z)<p(y_0-x_0).$$

The convexity of C implies that $(1/c)z+(1-1/c)x_0\in C$. Since $(1/c)z+(1-1/c)x_0=y\in\bar U$, it follows that

$$\begin{aligned} p(y_0-y) &= p\left[\frac{1}{c}(y_0-z)+\left(1-\frac{1}{c}\right)(y_0-x_0)\right]\\ &\leqslant \frac{1}{c}\,p(y_0-z)+\left(1-\frac{1}{c}\right)p(y_0-x_0)\\ &< p(y_0-x_0). \end{aligned}$$

This contradicts the choice of y_0. As a result, we have

$$p(y_0-x_0)\leqslant p(y_0-z)\quad\text{for all } z\in I_{\bar U}(x_0)\cap C.$$

The continuity of p further implies that

$$p(y_0-x_0)\leqslant p(y_0-z)\quad\text{for all } z\in\overline{I_{\bar U}(x_0)}\cap C.$$

Hence

$$0<p(y_0-x_0)=d_p(y_0,\bar U\cap C)=d_p(y_0,\overline{I_{\bar U}(x_0)}\cap C)$$

(here we have inequality since $x_0\in\overline{I_{\bar U}(x_0)}\cap C$). Suppose $x_0\in U$. Then $\overline{I_{\bar U}(x_0)}=E$, which implies $d_p(y_0,\overline{I_{\bar U}(x_0)}\cap C)=0$. Hence $x_0\in\partial_C(U)$. □

We omit the proof the following result as it can easily be derived using the same arguments as above.

Theorem 3.3. *Let C be a closed, convex subset of a Hausdorff locally space E with $0\in int(C)$. Suppose $s: C\to C$ is surjective and $F\in s\text{-}KKM(C,C,E)$ is a closed Φ-condensing map. Then there exist $x_0\in C$ and $y_0\in F(x_0)$ with*

$$p(y_0-x_0)=d_p(y_0,C)=d_p(y_0,\overline{I_C(x_0)}),$$

where p is the Minkowski functional of C in E. More precisely, either (i) *F has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial(C)$ and $y_0 \in F(x_0)$ with*

$$0 < p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, \overline{I_C(x_0)}).$$

As an immediate corollary, we have the following:

Corollary 3.4. *Let E be a normed space. Suppose $s : B_R \to B_R$ is surjective and $F \in s\text{-}KKM(B_R, B_R, E)$ is a closed Φ-condensing map. Then there exist $x_0 \in B_R$ and $y_0 \in F(x_0)$ with*

$$\|y_0 - x_0\| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)}).$$

More precisely, either (i) *F has a fixed point $x_0 \in B_R$, or* (ii). *there exist $x_0 \in \partial(B_R)$ and $y_0 \in F(x_0)$ with*

$$0 < \|y_0 - x_0\| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)}).$$

Proof. Since $p(x) = \|x\|/R$ is the Minkowski functional on B_R, we now apply Theorem 3.3. □

Remark 3.1. Corollary 3.4 extends Theorem 1 of Lin and Park [10] to the class s-KKM. The result in Lin [9] is also a special case of Corollary 3.4.

If s is the identity $\mathbf{1}_C$, then Theorem 3.3 reduces to the following result.

Corollary 3.5. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in int(C)$. Suppose $F \in KKM(C, E)$ is a closed Φ-condensing map. Then there exist $x_0 \in C$ and $y_0 \in F(x_0)$ with*

$$p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, \overline{I_C(x_0)}).$$

More precisely, either (i) *F has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial(C)$ and $y_0 \in F(x_0)$ with*

$$0 < p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, \overline{I_C(x_0)}).$$

Theorem 3.6. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $int(C) \neq \emptyset$. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, E)$ is a closed Φ-condensing map. Then for each $a \in int(C)$, there exist $x_0 = x_0(a) \in C$ and $y_0 \in F(x_0)$ with*

$$p_0(y_0 - x_0) = d_{p_0}(y_0, C) = d_{p_0}(y_0, \overline{I_C(x_0)}),$$

where p_0 *is the Minkowski functional of* $C-a$ *in* E. *More precisely, either* (i) F *has a fixed point* $x_0\in C$, *or* (ii) *there exist* $x_0\in\partial(C)$ *and* $y_0\in F(x_0)$ *with*

$$0< p_0(y_0-x_0)=d_{p_0}(y_0,C)=d_{p_0}(y_0,\overline{I_C(x_0)}).$$

Proof. Replacing C, F, and s by $\hat{C}:=C-a$, $\hat{F}:\hat{C}\to 2^E:\hat{F}(x-a)=F(x)-a$, and $\hat{s}:\hat{C}\to\hat{C}:\hat{s}(x-a)=s(x)-a$, respectively, we may assume that $0\in int(C)$. Now the result follows immediately from Theorem 3.3. □

Theorem 3.7. *Let* Φ *be either* α *or* χ *and* C *a nonempty, closed, convex subset of a Banach space* E *with* $0\in int(C)$. *Suppose* $s:C\to C$ *is surjective and* $F\in s$-*KKM* (C,C,E) *is a closed* 1-Φ-*contractive map. In addition, assume the following condition holds*:

$$\begin{cases} \text{if } \{x_n\}\subseteq C \text{ with } y_n\in F(x_n) \text{ for all } n \text{ and } x_n-r(y_n)\to 0\\ \quad \text{as } n\to\infty, \text{ then there exists an } x_0\in C \text{ with } x_0\in F(x_0).\end{cases}$$

Then there exist $x_0\in C$ *and* $y_0\in F(x_0)$ *with*

$$p(y_0-x_0)=d_p(y_0,C)=d_p(y_0,\overline{I_C(x_0)}),$$

where p *is the Minkowski functional of* C *in* E. *More precisely, either* (i) F *has a fixed point* $x_0\in C$, *or* (ii) *there exist* $x_0\in\partial(C)$ *and* $y_0\in F(x_0)$ *with*

$$0< p(y_0-x_0)=d_p(y_0,C)=d_p(y_0,\overline{I_C(x_0)}).$$

Proof. Let $r:E\to C$ be as defined above. Then r is continuous. Since $r(A)\subseteq\overline{co}(\{0\}\cup A)$ for each subset A of C, it follows that $\Phi(r(A))\leqslant\Phi(A)$ and so $G=r\circ F$ is 1-Φ-contractive. Also G is closed. By Remark 2.4, $G\in s\text{-}KKM(C,C,C)$. Now Theorem 3.1 implies that G has a fixed point x_0. Hence, as in Theorem 3.2, there exists y_0 with $y_0\in F(x_0)$ such that

$$p(y_0-x_0)=d_p(y_0,C)=d_p(y_0,\overline{I_C(x_0)}). \qquad \square$$

Using approximation results, we now obtain some fixed point theorems.

Theorem 3.8. *Let* C *be a closed, convex subset of a Hausdorff locally convex space* E *with* $0\in C$ *and* U *a convex open neighborhood of* 0. *Suppose* $s:\bar{U}\cap C\to\bar{U}\cap C$ *is surjective and* $F\in s\text{-}KKM(\bar{U}\cap C,\bar{U}\cap C,C)$ *is a closed* Φ-*condensing map. If* F *satisfies any one of the following conditions for any* $x\in\partial_C(U)\setminus F(x)$:

(i) *for each* $y\in F(x)$, $p(y-z)<p(y-x)$ *for some* $z\in\overline{I_{\bar{U}}(x)}\cap C$;
(ii) *for each* $y\in F(x)$, *there exist* λ *with* $|\lambda|<1$ *such that* $\lambda x+(1-\lambda)y\in\overline{I_{\bar{U}}(x)}\cap C$;
(iii) $F(x)\subseteq\overline{I_{\bar{U}}(x)}\cap C$;
(iv) $F(x)\cap\{\lambda x:\lambda>1\}=\emptyset$;
(v) *for each* $y\in F(x)$, $p(y-x)\neq p(y)-1$;

(vi) *for each* $y \in F(x)$, *there exist* $\alpha \in (1, \infty)$ *such that* $p^{\alpha}(y) - 1 \leqslant p^{\alpha}(y - x)$;
(vii) *for each* $y \in F(x)$, *there exist* $\beta \in (0, 1)$ *such that* $p^{\beta}(y) - 1 \geqslant p^{\beta}(y - x)$;

then F *has a fixed point.*

Proof. An application of Theorem 3.2 yields that either

(1) F has a fixed point in $\bar{U} \cap C$ or
(2) there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$0 < p(y_0) - 1 = p(y_0 - x_0) = d_p(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C),$$

where p is the Minkowski functional of U and f is the restriction of the continuous retraction r to C.

Suppose F satisfies condition (i). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (i), we have $p(y_0 - z) < p(y_0 - x_0)$ for some $z \in \overline{I_{\bar{U}}(x_0)} \cap C$. This contradicts the choice of x_0. Hence F has a fixed point.

Suppose F satisfies condition (ii). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (ii), there exists λ with $|\lambda| < 1$ such that $\lambda x_0 + (1 - \lambda) y_0 \in \overline{I_{\bar{U}}(x_0)} \cap C$. This implies that

$$\begin{aligned} p(y_0 - x_0) &\leqslant p(y_0 - (\lambda x_0 + (1 - \lambda) y_0)) = p(\lambda(y_0 - x_0)) \\ &= |\lambda| p(y_0 - x_0) < p(y_0 - x_0), \end{aligned}$$

which is a contradiction. Hence F has a fixed point.

The proof for condition (iii) is obvious.

Suppose F satisfies condition (iv). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (iv), $\lambda x_0 \neq y_0$ for each $\lambda > 1$. Notice that $x_0 = f(y_0) = y_0 / p(y_0)$ and so $y_0 = \lambda_0 x_0$ with $\lambda_0 = p(y_0) > 1$. Hence F has a fixed point.

Suppose F satisfies condition (v). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (v), $p(y_0 - x_0) \neq p(y_0) - 1$. But we have $p(y_0 - x_0) = p(y_0) - 1$. Hence F has a fixed point.

Suppose F satisfies condition (vi). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then condition (vi) implies that there exists $\alpha \in (1, \infty)$ with $p^{\alpha}(y_0) - 1 \leqslant p^{\alpha}(y_0 - x_0)$. Let $\lambda_0 = 1/p(y_0)$. Then $\lambda_0 \in (0, 1)$ and

$$\begin{aligned} \frac{(p(y_0) - 1)^{\alpha}}{p^{\alpha}(y_0)} = (1 - \lambda_0)^{\alpha} &< 1 - \lambda_0^{\alpha} \\ &\leqslant \frac{p^{\alpha}(y_0) - 1}{p^{\alpha}(y_0)} \\ &\leqslant \frac{p^{\alpha}(y_0 - x_0)}{p^{\alpha}(y_0)}. \end{aligned}$$

Thus $p(y_0 - x_0) > p(y_0) - 1$. This contradicts $p(y_0 - x_0) = p(y_0) - 1$.

Finally suppose F satisfies condition (vii). Then, as above (see the proof of (vi)), it can be seen that F has a fixed point. □

Remark 3.2. We have obtained a Leray–Schauder type result as an application of Theorem 3.2 (see Theorem 3.8(iv)). This was originally proved by Chang et al. [2]. Theorem 3.8 contains Corollary 4.1 of Chang et al. [2] as a special case.

Using Theorem 3.3 and following the same arguments as above, we get the following result.

Theorem 3.9. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in int(C)$. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C,C,E)$ is a closed Φ-condensing map. If F satisfies any one of the following conditions for any $x \in \partial(C) \setminus F(x)$:*

(i) *for each $y \in F(x)$, $p(y-z) < p(y-x)$ for some $z \in \overline{I_C(x)}$;*
(ii) *for each $y \in F(x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$;*
(iii) $F(x) \subseteq \overline{I_C(x)}$;
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) *for each $y \in F(x)$, $p(y-x) \neq p(y) - 1$;*
(vi) *for each $y \in F(x)$, there exist $\alpha \in (1,\infty)$ such that $p^{\alpha}(y) - 1 \leqslant p^{\alpha}(y-x)$;*
(vii) *for each $y \in F(x)$, there exist $\beta \in (0,1)$ such that $p^{\beta}(y) - 1 \geqslant p^{\beta}(y-x)$;*

then F has a fixed point.

Corollary 3.10. *Let E be a normed space. Suppose $s : B_R \to B_R$ is surjective and $F \in s\text{-}KKM(B_R,B_R,E)$ is a closed Φ-condensing map. If F satisfies any one of the following conditions for any $x \in \partial(B_R) \setminus F(x)$:*

(i) *for each $y \in F(x)$ $\|y-z\| < \|y-x\|$ for some $z \in \overline{I_{B_R}(x)}$;*
(ii) *for each $y \in F(x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_{B_R}(x)}$;*
(iii) $F(x) \subseteq \overline{I_{B_R}(x)}$;
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) *for each $y \in F(x)$, $\|y-x\| \neq \|y\| - R$;*
(vi) *for each $y \in F(x)$, there exist $\alpha \in (1,\infty)$ such that $\|y\|^{\alpha} - R \leqslant \|y-x\|^{\alpha}$;*
(vii) *for each $y \in F(x)$, there exist $\beta \in (0,1)$ such that $\|y\|^{\beta} - R \geqslant \|y-x\|^{\beta}$;*

then F has a fixed point.

Remark 3.3. Corollary 3.10 generalizes Theorem 2 of Lin and Park [10] as well as a result of Lin [9].

Corollary 3.11. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in int(C)$. Suppose $F \in KKM(C,E)$ is a closed Φ-condensing map. If F*

satisfies any one of the following conditions for any $x \in \partial(C) \setminus F(x)$:

(i) *for each* $y \in F(x)$, $p(y-z) < p(y-x)$ *for some* $z \in \overline{I_C(x)}$;
(ii) *for each* $y \in F(x)$, *there exist* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$;
(iii) $F(x) \subseteq \overline{I_C(x)}$;
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) *for each* $y \in F(x)$, $p(y-x) \neq p(y) - 1$;
(vi) *for each* $y \in F(x)$, *there exist* $\alpha \in (1,\infty)$ *such that* $p^{\alpha}(y) - 1 \leqslant p^{\alpha}(y-x)$;
(vii) *for each* $y \in F(x)$, *there exist* $\beta \in (0,1)$ *such that* $p^{\beta}(y) - 1 \geqslant p^{\beta}(y-x)$;

then F has a fixed point.

Applying Theorem 3.6, we have the following fixed point result which includes Corollary 4.2 of Chang et al. [2] as a special case.

Theorem 3.12. *Let C be a closed, convex subset of a Hausdorff locally convex space E with* $int(C) \neq \emptyset$. *Suppose* $s : C \to C$ *is surjective and* $F \in s\text{-}KKM(C,C,E)$ *is a closed* Φ*-condensing map. If F satisfies any one of the following conditions for any* $x \in \partial(C) \setminus F(x)$:

(i) *for each* $y \in F(x)$, $p_0(y-z) < p_0(y-x)$ *for some* $z \in \overline{I_C(x)}$;
(ii) *for each* $y \in F(x)$, *there exist* λ with $|\lambda| < 1$ *such that* $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$;
(iii) $F(x) \subseteq \overline{I_C(x)}$;
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) *for each* $y \in F(x)$, $p_0(y-x) \neq p_0(y) - 1$;
(vi) *for each* $y \in F(x)$, *there exist* $\alpha \in (1,\infty)$ *such that* $p_0^{\alpha}(y) - 1 \leqslant p_0^{\alpha}(y-x)$;
(vii) *for each* $y \in F(x)$, *there exist* $\beta \in (0,1)$ *such that* $p_0^{\beta}(y) - 1 \geqslant p_0^{\beta}(y-x)$;

then F has a fixed point.

We now state an application of Theorem 3.7.

Theorem 3.13. *Let* Φ *be either* α *or* χ *and C a nonempty, closed, convex subset of a Banach space E with* $0 \in int(C)$. *Suppose* $s : C \to C$ *is surjective and* $F \in s\text{-}KKM(C,C,E)$ *is a closed* 1-Φ*-contractive map. In addition, assume the following condition holds*:

$$\begin{cases} \text{if } \{x_n\} \subseteq C \text{ with } y_n \in F(x_n) \text{ for all } n \text{ and } x_n - r(y_n) \to 0 \\ \quad \text{as } n \to \infty, \text{ then there exists an } x_0 \in C \text{ with } x_0 \in F(x_0). \end{cases}$$

If F satisfies any one of the following conditions for any $x \in \partial(C) \setminus F(x)$:

(i) *for each* $y \in F(x)$, $p(y-z) < p(y-x)$ *for some* $z \in \overline{I_C(x)}$;
(ii) *for each* $y \in F(x)$, *there exist* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$;
(iii) $F(x) \subseteq \overline{I_C(x)}$;
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) *for each* $y \in F(x)$, $p(y-x) \neq p(y) - 1$;

(vi) *for each* $y \in F(x)$, *there exist* $\alpha \in (1, \infty)$ *such that* $p^\alpha(y) - 1 \leqslant p^\alpha(y - x)$;
(vii) *for each* $y \in F(x)$, *there exist* $\beta \in (0, 1)$ *such that* $p^\beta(y) - 1 \geqslant p^\beta(y - x)$;

then F *has a fixed point.*

Following the ideas above, it is possible to obtain other approximation and fixed point theorems in Hilbert spaces (here the retraction r is replaced by the proximity map). These theorems generalize Theorems 3 and 4 of Lin and Park [10].

Theorem 3.14. *Let* C *be a nonempty, closed, convex subset of a Hilbert space* H. *Suppose* $s : C \to C$ *is surjective and* $F \in s\text{-}KKM(C, C, H)$ *is a closed* Φ*-condensing map. Then there exist* x_0 *and* $y_0 \in F(x_0)$ *with*

$$\|y_0 - x_0\| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}),$$

where $\|.\|$ *is the norm induced by the inner product. More precisely, either* (i) F *has a fixed point* $x_0 \in C$, *or* (ii) *there exist* $x_0 \in \partial(C)$ *and* $y_0 \in F(x_0)$ *with*

$$0 < \|y_0 - x_0\| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}).$$

Proof. Let $r : H \to C$ be the proximity map. Then r is nonexpansive and so $G = r \circ F$ is Φ-condensing. By Remark 2.4, $G \in s\text{-}KKM(C, C, C)$. Now Lemma 2.1 guarantees that G has a fixed point i.e. there exists an $x_0 \in C$ with $x_0 \in G(x_0)$. Then there exists some $y_0 \in F(x_0)$ with $x_0 = r(y_0)$. Thus

$$\|x_0 - y_0\| = \|r(y_0) - y_0\| = \inf_{y \in C} \|y_0 - y\| = d(y_0, C).$$

As in the proof of Theorem 3.2, we can get

$$\|x_0 - y_0\| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}). \qquad \square$$

We only state the following result and leave the obvious details to the reader.

Theorem 3.15. *Let* C *be a nonempty, closed, convex subset of a Hilbert space* H. *Suppose* $s : C \to C$ *is surjective and* $F \in s\text{-}KKM(C, C, H)$ *is a closed* Φ*-condensing map. If* F *satisfies any one of the following conditions for any* $x \in \partial(C) \setminus F(x)$:

(i) *for each* $y \in F(x)$, $\|y - z\| < \|y - x\|$ *for some* $z \in \overline{I_C(x)}$;
(ii) *for each* $y \in F(x)$, *there exist* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1 - \lambda) y \in \overline{I_C(x)}$;
(iii) $F(x) \subseteq \overline{I_C(x)}$;

then F *has a fixed point.*

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